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Graceful switching in hybrid models

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***Abstract*— Classical oscillators, such as the ideal pendulum and the harmonic oscillator, can be modeled as switching systems. In these models, the switching takes place in a "graceful" fashion when the rate of displacement is zero. Such examples raise the question of whether hybrid models with such "graceful" switching can be modeled using ordinary differential equations, and of how to identify cases of hybrid systems which admit alternative classical descriptions in higher dimensions. Conversely, in a similar spirit, we discuss how a relaxation oscillator which consists of an integrator in feedback with an ideal relay-hysteresis can be approximated by a second order system.**

Consider the ideal pendulum depicted in Figure 1. This consists of a mass m suspended by a massless rod of length ℓ in a field with gravitational acceleration g. Applying Newton's laws this can be modeled by

$$\ddot{\theta} = -\frac{g}{\ell}\sin(\theta), \tag{1}$$

as a second-order system. Alternatitvely, if $\theta_{\max}$

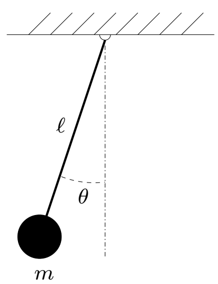

Fig. 1. Ideal pendulum.

represents the maximal angle with the vertical, then at any θ with $|\theta| \leq \theta_{\max}$, it holds that $\frac{1}{2}m(\ell\dot{\theta})^2 = mg\ell(\cos\theta - \cos\theta_{\max})$, and in a standard way we obtain the first integral of motion:

$$\dot{\theta}^2 = 2\frac{g}{\ell}(\cos\theta - \cos\theta_{\max}). \tag{2}$$

Differentiating (2) leads to (1). But (2) already describes the motion in the form of the first-order differential equation

$$\dot{\theta} = \pm\sqrt{2\frac{g}{\ell}(\cos\theta - \cos\theta_{\max})}. \tag{3}$$

Starting from (3) the solution can be expressed in terms of classical Jacobi elliptic functions (see [3, Chapter 17], [6]), but this approach will not concern us herein.

We wish to interpret (3) as defining the hybrid (switching) system:

$$\dot{\theta} = u\sqrt{2\frac{g}{\ell}(\cos\theta - \cos\theta_{\max})}, \tag{4}$$

where:

a) the state $(\theta, u) \in \mathbb{R} \times \{+1, -1\}$,
b) $u(t)$ remains constant as long as $|\theta(t)| \neq \theta_{\max}$,
c) when $|\theta(t_1)| = \theta_{\max}$, then $u(t_1) \to -u(t_1)$, and
d) on any interval $[t_1, t_1 + \epsilon)$, for small $\epsilon > 0$, select the unique solution $\theta(t)$ of (4) which is not constant.

The above rules specify uniquely the dynamical evolution of a hybrid (switching) system, where switching occurs in a "graceful" manner, when the rate $\dot{\theta}$ is zero. The reason for rule *d)* stems from the fact that (4) is not Lipschitz-continuous when $|\theta(t_1)| = \theta_{\max}$, and admits many solutions. All solutions but one are constant on intervals $[t_1, t_1 + \epsilon)$ for sufficiently small $\epsilon > 0$.

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A similar description is valid for the linear harmonic oscillator

$$\ddot{x} = -x, \tag{5}$$

which admits a first integral of motion $\dot{x}^2 = 1 - x^2$ and hence, a description as a switching system

$$\dot{x} = u\sqrt{1 - x^2}, \tag{6}$$

with $u \in \{+1, -1\}$ and a similar set of rules for switching as above. Switching for this system as well takes place in a graceful manner, when $\dot{x} = 0$. This type of transition occurs when the trajectory of the system passes onto different branches of a Riemann surface. Naturally, by differentiating (6) we can eliminate u and recover (5).

The above examples raise the question as to how to identify cases of hybrid systems which admit alternative classical descriptions in higher dimensions. It appears that cases where switching occurs only when the rate of change of the state vector is zero, fall in this category. Conversely, these examples raise the possibility that a switching system may be approximated by one with this type of a property (i.e., "graceful" switching) and then converted to a smooth ordinary differential equation. We now amplify this idea by referring to the ordinary play (i.e., the ideal relay hysteresis element) when in feedback with negative integrator.

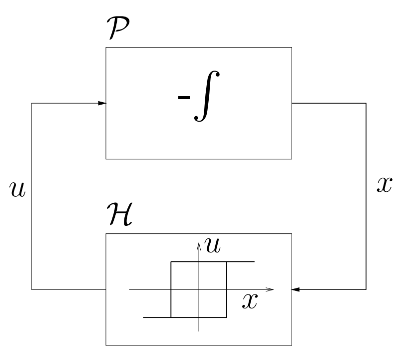

Fig. 2. Relay-relaxation oscillator

A relay-hysteresis $\mathcal{H}(\cdot)$ is defined for a continuous input $x(t)$ (see e.g., [5, p. 66]). For simplicity we may assume that $u(0) = 1$, $x(0) = 0$ and that $|x| \leq 1$. Then, the output $u(t)$ takes values from the set $\{-1, +1\}$ according to the following set of rules:

i) $u(t) \rightarrow -u(t)$ when $|x(t)| = 1$ and $x(t)u(t) < 0$,
ii) $u(t)$ stays constant otherwise.

The analysis of switching systems with relays requires care due to the discontinuous nature of the outputs of such elements. The well-posedness of feedback systems with such discontinuous elements has been the subject of several investigations [2], [7], [8]. A typical simplification calls for avoiding arbitrarily fast switching, in which case existence and uniqueness of solutions can be ensured by integrating dynamic elements over successive intervals where the output of the relay is constant.

An alternative approach can be based on replicating the earlier idea and approximating the discontinuous relay with the "gentler" nonlinear element $f(x) = \pm \sqrt[2k]{1 - x_{\mathrm{mod}2}^{2k}}$ over the whole axis, for $k \in \mathbb{N}$, a suitable choice of sign, and $x_{\mathrm{mod}2}$

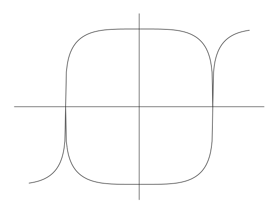

Fig. 3. Continuous approximation of relay hysteresis.

chosen to take values in $[-1, 1]$ —this is shown in Figure 3. In this way the feedback system of Figure 2 is approximated by

$$\dot{x} = u \sqrt[2k]{1 - x^{2k}}, \tag{7}$$

In particular, u switches between $+1$ and -1 as before, but only when the output of the nonlinear element $f(x)$ is zero. Then we eliminate the switch altogether by differentiating (7) to obtain the second-order ordinary differential equation

$$\begin{aligned} \ddot{x} &= u\left(\frac{\partial}{\partial x}\sqrt[2k]{1-x^{2k}}\right)\dot{x} \\ &= -x\frac{1}{\left(\frac{1}{x^k}-x^k\right)^{\frac{k-1}{k}}}. \end{aligned} \tag{8}$$

The response of (8) is shown in Figure 4 for

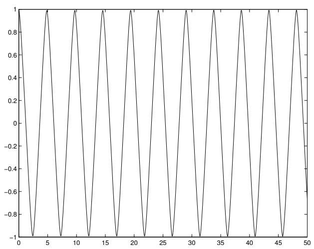

Fig. 4. Response of (8) for $k = 2$.

$k = 2$. This is the exact response of the feedback system in Figure 2 when replacing the ideal relay hysteresis with the nonlinearity shown in Figure 3. Thus, switching has been replaced by an ordinary differential equation, albeit a rather stiff one. As expected, this gives a fairly good approximation of the response of the system in Figure 2 with the ideal relay hysteresis in place.

To recap, when in hybrid systems switching takes place in a graceful manner, it may be beneficial to model and analyze such systems by using ordinary differential equations instead. This suggests a possible route for approximating more general switching systems.

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